CHAPTER 4

TRIGONOMETRIC ANALYSIS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Use the rectangular coordinate system to determine the algebraic signs and the values of the trigonometric functions and to locate and define the trigonometric functions.

2. Relate any angle in standard position to its reference angle.

3. Determine the trigonometric functions of an angle in any quadrant, of negative angles, of coterminal angles, of frequently used angles, and of quadrantal angles.

4. Express the trigonometric functions of an angle in terms of their complement.

5. Recognize characteristics of the graphs of the sine, cosine, and tangent functions.

INTRODUCTION

This chapter is a continuation of the broad topic of trigonometry introduced in chapter 3. The topic is expanded in this chapter to allow analysis of angles greater than 90°. The chapter is extended as a foundation for analysis of the generalized angle; that is, an angle of any number of degrees. Additionally, the chapter introduces the concept of both positive and negative angles.

RECTANGULAR COORDINATE SYSTEM

The rectangular, or Cartesian, coordinate system introduced in *Mathematics*, Volume 1, was used in solving equations; in this chapter it is used to analyze the generalized angle. The following is a brief review of the rectangular coordinate system:

1. The vertical axis (Y axis in fig. 4-1) is considered positive above the origin and negative below the origin.

2. The horizontal axis (X axis in fig. 4-1) is positive to the right of the origin and negative to the left of the origin.

3. A point, $P(x,y)$, anywhere in a rectangular coordinate system may be located by two numbers. The value of x is called the *abscissa*. The value of y is called the *ordinate*. The abscissa and ordinate of a point are its *coordinates*.

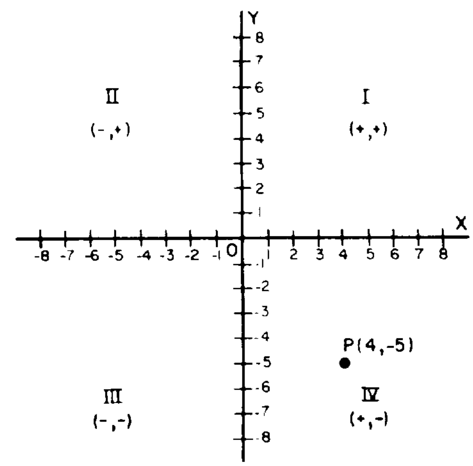

Figure 4-1.—Rectangular coordinate system.

4. In notation used to locate points, the coordinates are conventionally placed in parentheses and separated with a comma, with the abscissa always written first. The general form of this notation is $P(x,y)$. Thus, point P in figure 4-1 would have the notation $P(4, -5)$.

5. The quadrants are numbered in the manner described in chapter 3 of this course (shown as Roman numerals in figure 4-1).

6. The x coordinate is positive in the first (I) and fourth (IV) quadrants and negative in the second (II) and third (III) quadrants. The y coordinate is positive in the first and second quadrants and negative in the third and fourth quadrants. The signs of the coordinates are shown in parentheses in figure 4-1. The algebraic signs of the coordinates of a point are used in this chapter for determining the algebraic signs of trigonometric functions.

ANGLES IN STANDARD POSITION

To construct an *angle in standard position*, first lay out a rectangular coordinate system. Then draw the angle, θ, so that its vertex is at the origin of the coordinate system and its initial or original side is lying along the positive X axis as shown in figure 4-2. The terminal or final side of the angle will lie in any of the quadrants or on one of the axes separating the quadrants. When the terminal side falls on an axis, the angle is called a *quadrantal angle*, which will be discussed later in this chapter. In figure 4-2 the terminal side lies in quadrant II.

The quadrant in which an angle lies is determined by the terminal side. When an angle is placed in standard position, the angle is said to lie in the quadrant containing the terminal side. For example, the negative angle, θ, shown in standard position in figure 4-3, is said to lie in the second quadrant.

When two or more angles in standard position have their terminal sides located at the same position, they are said to be *coterminal*. If θ is any general angle, then θ plus or minus an integral multiple of 360° yields a coterminal angle.

For example, the angles θ, ϕ, and α in figure 4-4 are said to be coterminal angles. If

$$\theta = 45^\circ$$

then

$$\begin{aligned} \phi &= \theta - 360^\circ \\ &= 45^\circ - 360^\circ \\ &= -315^\circ \end{aligned}$$

and

$$\begin{aligned} \alpha &= \theta + 360^\circ \\ &= 45^\circ + 360^\circ \\ &= 405^\circ \end{aligned}$$

The relationship of coterminal angles can be stated in a general form. *For any angle θ measured in degrees, any angle ϕ coterminal with θ can be found by*

$$\phi = \theta + n(360^\circ)$$

where n is any integer (positive, negative, or zero); that is,

$$n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

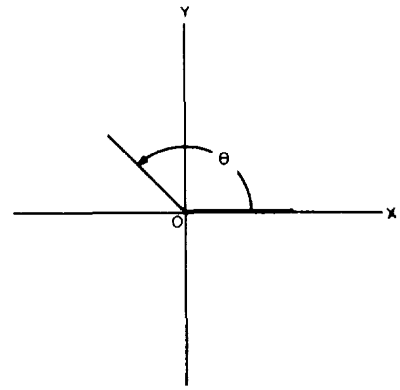

Figure 4-2.—Angle in standard position.

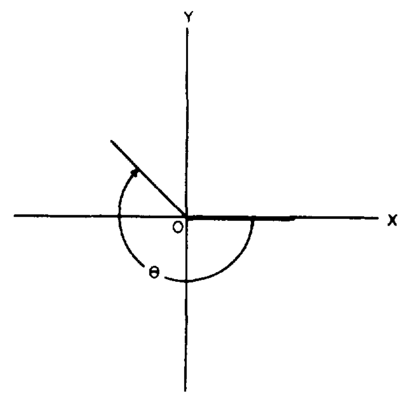

Figure 4-3.—Negative angle in quadrant II.

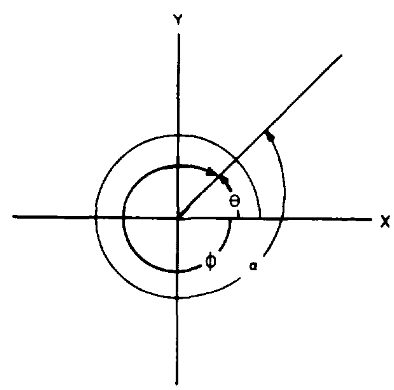

Figure 4-4.—Coterminal angles.

The principle of coterminal angles is used in developing other trigonometric relationships and other phases of trigonometric analysis. An expansion of this principle, discussed later in this chapter, states that the trigonometric functions of coterminal angles have the same value.

PRACTICE PROBLEMS:

Determine whether or not the following sets of angles are coterminal:

1. $60°$, $-300°$, $420°$

2. $0°$, $360°$, $180°$

3. $45°$, $-45°$, $345°$

4. $735°$, $-345°$, $-705°$

ANSWERS:

1. Coterminal

2. Not coterminal

3. Not coterminal

4. Coterminal

DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

So far, the trigonometric functions have been defined as follows:

1. By labeling the sides of a right triangle x, y, and r.
2. By naming the sides of a right triangle adjacent, opposite, and hypotenuse.

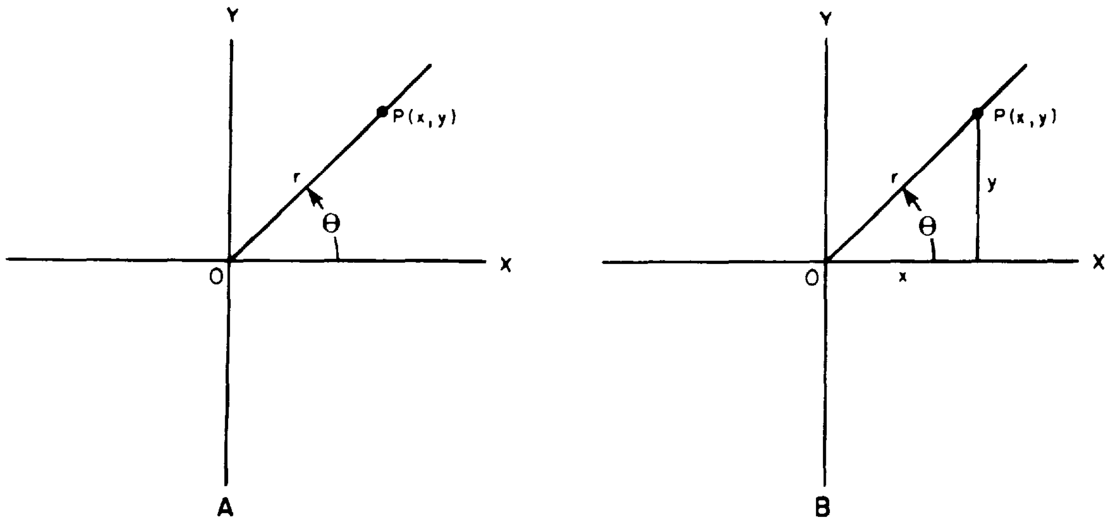

Figure 4-5.—Functions of general angles.

In this chapter we will introduce a third set of definitions using the nomenclature of the coordinate system. Note that each definition defines the same functions using different terminology.

To arrive at the third set of definitions, construct an angle in standard position on a coordinate system as shown in figure 4-5, view A. Choose point $P(x,y)$ on the final position of the radius vector. Distance OP is denoted by the positive number r for the length of the radius.

By constructing a right triangle using $P(x,y)$ and r, as in figure 4-5, view B, the six trigonometric functions are classified as follows:

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{length of radius}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{length of radius}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{length of radius}}{\text{abscissa}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{length of radius}}{\text{ordinate}}$$

The value of each function is dependent on angle θ and not on the selection of point $P(x,y)$. If a different point were chosen, the length of r, as well as the values of the x and y coordinates, would change proportionally, but the ratios would be unchanged.

EXAMPLE: Find the sine and cosine of angle θ in figure 4-5, view A, for the point $P(3,4)$.

SOLUTION: To determine the sine and cosine of θ, we must find the value of r. Since the values of the x and y coordinates correspond to the lengths of the sides x and y in figure 4-5, view B, we can determine the length of r by using the Pythagorean theorem or by recalling from *Mathematics*, Volume 1, the 3-4-5 triangle. In either case, the length of r is 5 units. Hence,

$$\sin \theta = \frac{\text{ordinate}}{\text{length of radius}}$$

$$= \frac{4}{5}$$

and

$$\cos \theta = \frac{\text{abscissa}}{\text{length of radius}}$$

$$= \frac{3}{5}$$

NOTE: For the remainder of this chapter, all angles are understood to be in standard position, unless otherwise stated.

PRACTICE PROBLEMS:

Find the sine, cosine, and tangent of the angles whose radius vectors pass through the following points:

1. $P(5,12)$
2. $P(1,1)$
3. $P(1, \sqrt{3})$
4. $P(3,2)$

ANSWERS:

1. $\sin\theta = 12/13$

   $\cos\theta = 5/13$

   $\tan\theta = 12/5$

2. $\sin\theta = 1/\sqrt{2} = \sqrt{2}/2$

   $\cos\theta = 1/\sqrt{2} = \sqrt{2}/2$

   $\tan\theta = 1/1 = 1$

3. $\sin\theta = \sqrt{3}/2$

   $\cos\theta = 1/2$

   $\tan\theta = \sqrt{3}/1 = \sqrt{3}$

4. $\sin\theta = 2/\sqrt{13} = 2\sqrt{13}/13$

   $\cos\theta = 3/\sqrt{13} = 3\sqrt{13}/13$

   $\tan\theta = 2/3$

QUADRANT SYSTEM

The quadrants formed in the rectangular coordinate system are used to determine the algebraic signs of the trigonometric functions. The quadrants in figure 4-6 show the algebraic signs of the trigonometric functions in the various quadrants.

In the first quadrant the abscissa and ordinate are always positive. The radius vector is always taken as positive. Therefore, *all the trigonometric ratios are positive for angles in the first quadrant. For angles in the second quadrant, only the ratios involving the ordinate and the radius vector are positive. These are the sine and cosecant ratios. For angles in the third quadrant, where the ordinate and abscissa are both negative, only the ratios involving the abscissa and the ordinate are positive.*

II	I
sin θ = +/+ = +	sin θ = +/+ = +
cos θ = -/+ = -	cos θ = +/+ = +
tan θ = +/- = -	tan θ = +/+ = +
cot θ = -/+ = -	cot θ = +/+ = +
sec θ = +/- = -	sec θ = +/+ = +
csc θ = +/+ = +	csc θ = +/+ = +
III	**IV**
sin θ = -/+ = -	sin θ = -/+ = -
cos θ = -/+ = -	cos θ = +/+ = +
tan θ = -/- = +	tan θ = -/+ = -
cot θ = -/- = +	cot θ = +/- = -
sec θ = +/- = -	sec θ = +/+ = +
csc θ = +/- = -	csc θ = +/- = -

Figure 4-6.—Signs of functions.

These are the tangent and cotangent ratios. For angles in the fourth quadrant, ratios involving the radius vector and the abscissa are positive. These are the cosine and the secant ratios.

NOTE: In each quadrant the sine and cosecant have the same sign, the cosine and the secant have the same sign, and the tangent and cotangent have the same sign.

The last group of practice problems involved angles in the first quadrant only, where all of the functions were positive. When an angle lies in one of the other quadrants, the trigonometric functions may be positive or negative.

EXAMPLE: Find all of the trigonometric functions of θ if $\tan \theta = 5/12$, $\sin \theta < 0$, and $r = 13$.

SOLUTION: Reference to figure 4-6 shows that an angle with a positive tangent and a negative sine can only occur in the third quadrant. The point in the third quadrant has coordinates $(-12, -5)$. (See fig. 4-7)

We can now read the trigonometric ratios from the figure:

$$\sin \theta = \frac{\text{ordinate}}{\text{length of radius}} = \frac{-5}{13}$$

$$\cos \theta = \frac{\text{abscissa}}{\text{length of radius}} = \frac{-12}{13}$$

$$\tan \theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{-5}{-12} = \frac{5}{12}$$

$$\cot \theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{-12}{-5} = \frac{12}{5}$$

$$\sec \theta = \frac{\text{length of radius}}{\text{abscissa}} = \frac{13}{-12} = \frac{-13}{12}$$

$$\csc \theta = \frac{\text{length of radius}}{\text{ordinate}} = \frac{13}{-5} = \frac{-13}{5}$$

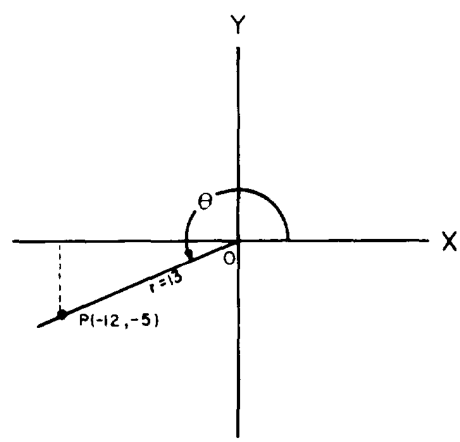

Figure 4-7.—Finding the trigonometric functions for a third-quadrant angle.

EXAMPLE: Find all of the trigonometric functions of θ if $\csc \theta = -17/15$ and $\cos \theta < 0$.

SOLUTION: The cosecant is negative in the same quadrants as the sine; that is, quadrants III and IV. The cosine is negative in quadrants II and III. Therefore, the cosecant and cosine are both negative in quadrant III. (Refer to fig. 4-6.) The ordinate in the third quadrant is -15 and the radius is 17.

NOTE: The fraction $-17/15$ indicates that either the numerator or denominator is negative, but not both. In this case, we know that the ordinate (denominator) is negative since the radius (numerator) is always positive.

From the Pythagorean theorem the abscissa in the third quadrant is

$$x^2 = r^2 - y^2$$

$$= (17)^2 - (-15)^2$$

$$= 289 - 225$$

$$= 64$$

$$x = -8$$

Therefore, referring to figure 4-8, the six trigonometric functions are as follows:

$$\sin \theta = -15/17$$

$$\cos \theta = -8/17$$

$$\tan \theta = -15/-8 = 15/8$$

$$\cot \theta = -8/-15 = 8/15$$

$$\sec \theta = 17/-8 = -17/8$$

$$\csc \theta = 17/-15 = -17/15$$

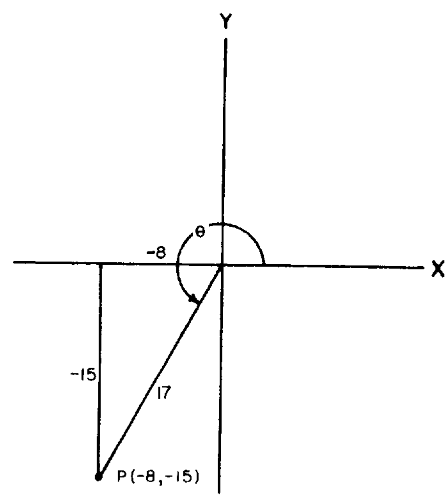

Figure 4-8.—Construction of a triangle in quadrant 3.

EXAMPLE: If $\sec \theta = -25/24$ and $\tan \theta = -7/24$, find the other four trigonometric ratios of θ.

SOLUTION: The secant and tangent are both negative in the second quadrant. In the second quadrant the abscissa is -24, the ordinate is 7, and the radius is 25 (refer to fig. 4-9); so,

$$\sin \theta = 7/25$$

$$\cos \theta = -24/25$$

$$\cot \theta = -24/7$$

$$\csc \theta = 25/7$$

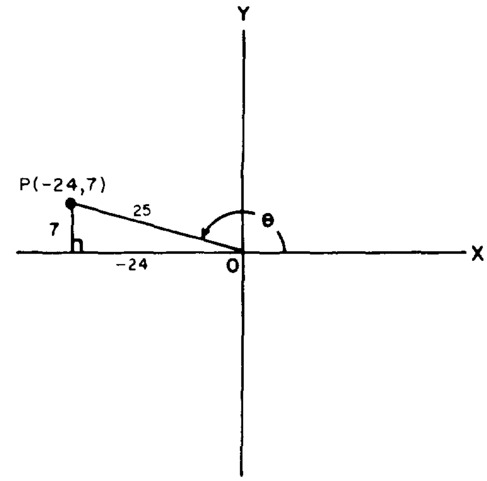

Figure 4-9.—Construction of a triangle in quadrant 2.

PRACTICE PROBLEMS:

Without using tables, find the six trigonometric functions of θ under the following conditions:

1. $\tan \theta = 3/4$, $r = 5$, and θ is not in the first quadrant.
2. $\tan \theta = -21/20$, $r = 29$, and $\cos \theta > 0$.
3. $\cos = -3/5$ and $\cot \theta = 3/4$.
4. $\tan \theta = -8/15$ and $\csc \theta$ is positive.

Indicate the quadrant in which the terminal side of θ lies for the following conditions:

5. $\sin \theta > 0$ and $\cos \theta < 0$
6. $\cos \theta < 0$ and $\csc \theta < 0$
7. $\sec \theta > 0$ and $\cot \theta < 0$

ANSWERS:

1. $\sin \theta = -3/5$

   $\cos \theta = -4/5$

   $\tan \theta = -3/-4 = 3/4$

   $\cot \theta = -4/-3 = 4/3$

   $\sec \theta = 5/-4 = -5/4$

   $\csc \theta = 5/-3 = -5/3$

2. $\sin\theta = -21/29$

$\cos\theta = 20/29$

$\tan\theta = -21/20$

$\cot\theta = 20/-21 = -20/21$

$\sec\theta = 29/20$

$\csc\theta = 29/-21 = -29/21$

3. $\sin\theta = -4/5$

$\cos\theta = -3/5$

$\tan\theta = -4/-3 = 4/3$

$\cot\theta = -3/-4 = 3/4$

$\sec\theta = 5/-3 = -5/3$

$\csc\theta = 5/-4 = -5/4$

4. $\sin\theta = 8/17$

$\cos\theta = -15/17$

$\tan\theta = 8/-15 = -8/15$

$\cot\theta = -15/8$

$\sec\theta = 17/-15 = -17/15$

$\csc\theta = 17/8$

5. 2

6. 3

7. 4

REFERENCE ANGLE

The *reference angle*, θ', for any angle, θ, in standard position is the smallest positive angle between the radius vector of θ and the X axis, such that $0° \leq \theta' \leq 90°$. In general, the reference angle for θ is

$$\theta' = n(180°) \pm \theta$$

where n is any integer. Expressed in an equivalent form

$$\theta' = n\pi \pm \theta$$

where again n is any integer and $0 \leq \theta' \leq \pi/2$.

Refer to figure 4-10. If P is any point on the radius vector, a perpendicular from P to the point A on the X axis forms a right triangle with sides OA, AP, and OP. We call this triangle the *reference triangle*. The relationship between θ, θ', and the reference triangle in each quadrant is shown in figure 4-10.

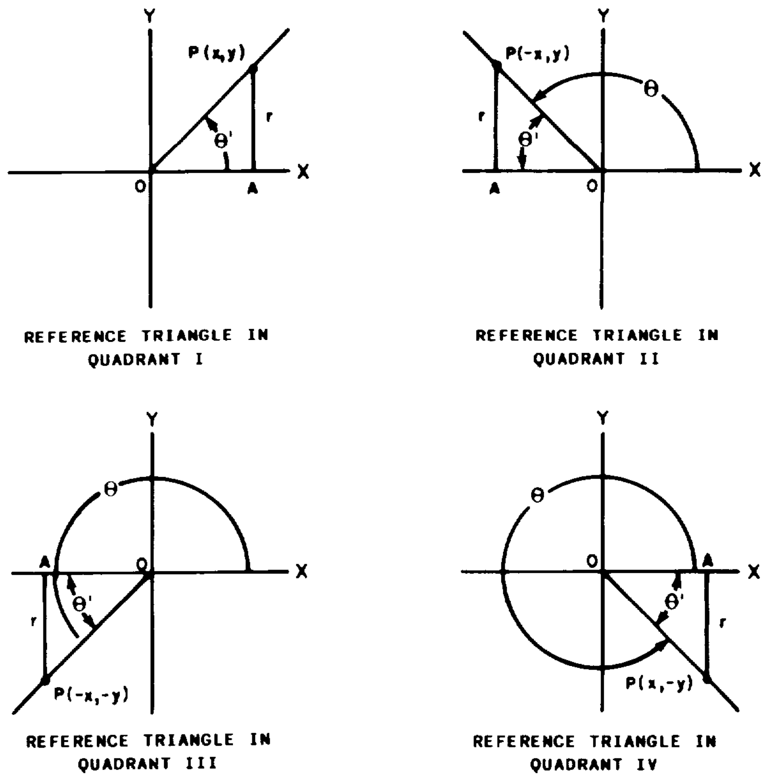

Figure 4-10.—Reference triangles in each quadrant.

FUNCTIONS OF ANGLES IN ANY QUADRANT

In addition to the reference triangle, formulas are used for determining the signs of the functions at any angle. These are called *reduction formulas*. This section shows the geometrical development of some of the most commonly used reduction formulas. In general, reduction formulas provide a means of reducing the functions of any angle to an equivalent expression for the function in terms of a positive acute angle, θ. The reduction formulas can be used in the solution of some trigonometric identities and in other applications requiring analysis of trigonometric functions.

The function of θ and the reduction formulas of the functions of $180° - \theta$, $180° + \theta$, and $360° - \theta$ are summarized in the following paragraphs according to their respective quadrants.

QUADRANT I

Any angle in the first quadrant can be represented by θ; that is,

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$

$$\sec \theta = \frac{r}{x}$$

$$\csc \theta = \frac{r}{y}$$

QUADRANT II

Any angle in the second quadrant can be represented by $180° - \theta$; that is,

$$\sin (180° - \theta) = \frac{y}{r} = \sin \theta$$

$$\cos (180° - \theta) = -\frac{x}{r} = -\cos \theta$$

$$\tan (180° - \theta) = -\frac{y}{x} = -\tan \theta$$

$$\cot (180° - \theta) = -\frac{x}{y} = -\cot \theta$$

$$\sec (180° - \theta) = -\frac{r}{x} = -\sec \theta$$

$$\csc (180° - \theta) = \frac{r}{y} = \csc \theta$$

EXAMPLE: Use a reduction formula and appendix III to find the cotangent of 112°.

SOLUTION: Since 112° is in the second quadrant, where

$$\cot (180° - \theta) = -\cot \theta$$

then

$$\cot 112° = \cot (180° - 68°)$$

$$= -\cot 68°$$

$$= -0.40403$$

QUADRANT III

Any angle in the third quadrant can be represented by $180° + \theta$; that is,

$$\sin (180° + \theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (180° + \theta) = -\frac{x}{r} = -\cos \theta$$

$$\tan (180° + \theta) = \frac{y}{x} = \tan \theta$$

$$\cot (180° + \theta) = \frac{x}{y} = \cot \theta$$

$$\sec (180° + \theta) = -\frac{r}{x} = -\sec \theta$$

$$\csc (180° + \theta) = -\frac{r}{y} = -\csc \theta$$

EXAMPLE: Use a reduction formula and appendix II to find the sine of 220°.

SOLUTION: Since 220° is in the third quadrant, where

$$\sin (180° + \theta) = -\sin \theta$$

then

$$\sin 220^\circ = \sin (180^\circ + 40^\circ)$$

$$= -\sin 40^\circ$$

$$= -0.64279$$

QUADRANT IV

Any angle in the fourth quadrant can be represented by $360^\circ - \theta$; that is,

$$\sin (360^\circ - \theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (360^\circ - \theta) = \frac{x}{r} = \cos \theta$$

$$\tan (360^\circ - \theta) = -\frac{y}{x} = -\tan \theta$$

$$\cot (360^\circ - \theta) = -\frac{x}{y} = -\cot \theta$$

$$\sec (360^\circ - \theta) = \frac{r}{x} = \sec \theta$$

$$\csc (360^\circ - \theta) = -\frac{r}{y} = -\csc \theta$$

EXAMPLE: Find cos 324°.

SOLUTION: Since

$$\cos (360^\circ - \theta) = \cos \theta$$

then

$$\cos 324^\circ = \cos (360^\circ - 36^\circ)$$

$$= \cos 36^\circ$$

$$= 0.80902$$

FUNCTIONS OF NEGATIVE ANGLES

The following relationships enable us to change a function with a negative angle into the same function with a positive angle:

$$\sin(-\theta) = -\frac{y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = -\frac{y}{x} = -\tan\theta$$

$$\cot(-\theta) = -\frac{x}{y} = -\cot\theta$$

$$\sec(-\theta) = \frac{r}{x} = \sec\theta$$

$$\csc(-\theta) = -\frac{r}{y} = -\csc\theta$$

EXAMPLE: Find $\tan(-350°)$.

SOLUTION: Since

$$\tan(-\theta) = -\tan\theta$$

then

$$\tan(-350°) = -\tan 350°$$

and

$$\begin{aligned} -\tan 350° &= -\tan(360° - 10°) \\ &= -(-\tan 10°) \\ &= 0.17633 \end{aligned}$$

FUNCTIONS OF COTERMINAL ANGLES

For a *coterminal angle* in the form of

$$\theta' = n(360°) + \theta$$

where n is any integer θ and is an integral multiple of θ', the trigonometric functions of θ' are equal to those of θ. In other words, θ is the remainder obtained by dividing θ' by 360, and n is the number of times 360 will divide into θ'. Thus, we can find the ratios of a coterminal angle greater than 360° by dividing θ' by 360 and finding the functions of the remainder.

EXAMPLE: Find the cosine of $-2{,}080°$. (Refer to fig. 4-11.)

SOLUTION: Divide 2,080 by 360.

$$\begin{array}{r|l} & 5 \\ \hline 360 & 2{,}080 \\ & \underline{1{,}800} \\ & 280 \end{array}$$

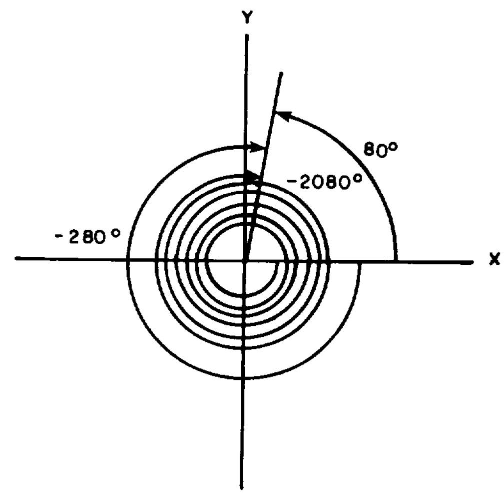

Figure 4-11.—Coterminal angles of $-2{,}080°$.

So,

$$\cos(-2{,}080°) = \cos(-280°)$$

and

$$\begin{aligned} \cos(-280°) &= \cos(280°) \\ &= \cos(360° - 80°) \\ &= \cos 80° \\ &= 0.17365 \end{aligned}$$

PRACTICE PROBLEMS:

Use reduction formulas and appendixes II and III to find the values of the sine, cosine, and tangent of θ given the following angles:

1. 137°
2. 214°
3. 325°
4. −70°
5. 1,554°

ANSWERS:

1. $\sin 137° = \sin 43° = 0.68200$

   $\cos 137° = -\cos 43° = -0.73135$

   $\tan 137° = -\tan 43° = -0.93252$

2. $\sin 214° = -\sin 34° = -0.55919$

   $\cos 214° = -\cos 34° = -0.82904$

   $\tan 214° = \tan 34° = 0.67451$

3. $\sin 325° = -\sin 35° = -0.57358$

   $\cos 325° = \cos 35° = 0.81915$

   $\tan 325° = -\tan 35° = -0.70021$

4. $\sin(-70°) = -\sin 70° = -0.93969$

   $\cos(-70°) = \cos 70° = 0.34202$

   $\tan(-70°) = -\tan 70° = -2.74748$

5. $\sin 1{,}554° = \sin 114° = \sin 66° = 0.91355$

   $\cos 1{,}554° = \cos 114° = -\cos 66° = -0.40674$

   $\tan 1{,}554° = \tan 114° = -\tan 66° = -2.24604$

COFUNCTIONS AND COMPLEMENTARY ANGLES

Complementary angles are angles whose sum is 90°. Two trigonometric functions that have equal values for complementary angles are called *cofunctions*.

Inspect the triangle in figure 4-12. We will compare the six trigonometric functions of θ with the six trigonometric functions of $90° - \theta$.

Functions of θ	Functions of $90° - \theta$
$\sin \theta = \frac{y}{r}$	$\cos (90° - \theta) = \frac{y}{r}$
$\cos \theta = \frac{x}{r}$	$\sin (90° - \theta) = \frac{x}{r}$
$\tan \theta = \frac{y}{x}$	$\cot (90° - \theta) = \frac{y}{x}$
$\cot \theta = \frac{x}{y}$	$\tan (90° - \theta) = \frac{x}{y}$
$\sec \theta = \frac{r}{x}$	$\csc (90° - \theta) = \frac{r}{x}$
$\csc \theta = \frac{r}{y}$	$\sec (90° - \theta) = \frac{r}{y}$

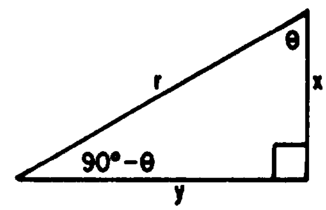

Figure 4-12.—Complementary angles.

We see from the above relationships that

$$\sin \theta = \cos (90° - \theta)$$

$$\cos \theta = \sin (90° - \theta)$$

$$\tan \theta = \cot (90° - \theta)$$

$$\cot \theta = \tan (90° - \theta)$$

$$\sec \theta = \csc (90° - \theta)$$

$$\csc \theta = \sec (90° - \theta)$$

Hence, a trigonometric function of an angle is equal to the confunction of its complement.

NOTE: These relationships may explain to you how the cosine, cotangent, and cosecant functions received their names.

The confunction principle accounts for the format of the tables of trigonometric functions in appendixes II and III. For example, in appendix II

$$\sin 21° 30' = 0.36650$$

and

$$\cos 68^\circ 30' = 0.36650$$

Notice that

$$21^\circ 30' + 68^\circ 30' = 90^\circ$$

PRACTICE PROBLEMS:

Express the following as a function of the complementary angle:

1. $\sin 27^\circ$
2. $\tan 38^\circ 17'$
3. $\csc 41^\circ$
4. $\cos 16^\circ 30' 22''$
5. $\sec 79^\circ 37' 16''$
6. $\cos 56^\circ$
7. $\cot 48^\circ$

ANSWERS:

1. $\cos 63^\circ$
2. $\cot 51^\circ 43'$
3. $\sec 49^\circ$
4. $\sin 73^\circ 29' 38''$
5. $\csc 10^\circ 22' 44''$
6. $\sin 34^\circ$
7. $\tan 42^\circ$

SPECIAL ANGLES

Two groups of angles are discussed in this section. The first group of angles is considered because the angles can be determined geometrically and are used frequently in problem solving. The second group is considered because the radius vectors of the angles fall on one of the coordinate axes, not in one of the quadrants.

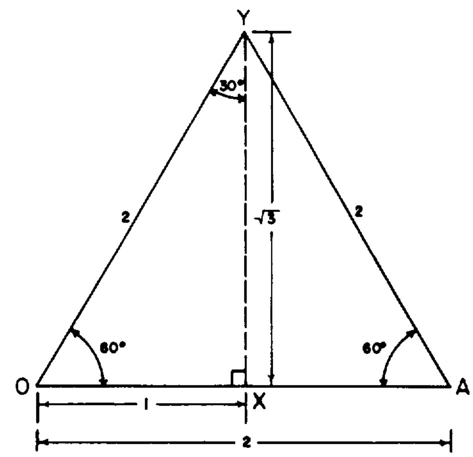

Figure 4-13.—Geometrical construction of 30° and 60° right triangles.

FREQUENTLY USED ANGLES

As stated previously, the approximate values of the trigonometric functions for any angle can be read directly from tables or can be determined from tables by the use of the principles stated in this text. However, certain frequently used simple angles exist for which the exact function values are often used because these exact values can easily be determined geometrically. In the following paragraphs the geometrical determination of these functions is shown.

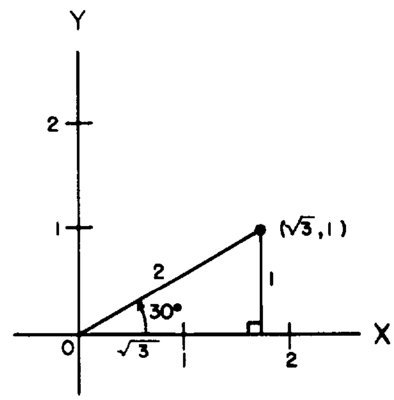

Figure 4-14.—30° reference triangle.

30°-60° Angles

The trigonometric functions of 30° and 60° can be determined geometrically. Construct an equilateral triangle with side lengths of 2 units, such as triangle *OYA* in figure 4-13. (The functions to be determined are not dependent on the lengths of the sides being 2 units; this size was selected for convenience.)

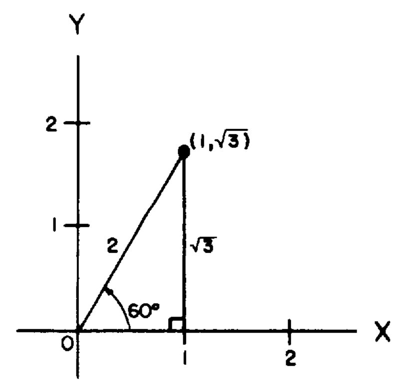

Figure 4-15.—60° reference triangle.

Drop a perpendicular from angle *Y* to the base of the triangle at point *X*. The right triangles *YXO* and *YXA* are formed by the perpendicular, which also bisects angle *Y* forming a 30° angle. Moreover, since side *OA* is 2 units long, then *OX* is 1 unit long and *YX* is $\sqrt{3}$ units long (using the Pythagorean theorem).

Figures 4-14 and 4-15 show a 30° and a 60° reference triangle, respectively. From these figures we can determine the trigonometric ratios of 30° and 60°, which are summarized in table 4-1.

Table 4-1.—Trigonometric Functions of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$

NOTE: The values of the confunctions interchange since 30° and 60° are complementary angles. An easy way to recall the values of the functions of right triangles with 30° and 60° complementary angles is to remember that the ratio of the sides is always 1, 2, and $\sqrt{3}$, where the largest side value represents the length of the hypotenuse.

EXAMPLE: Find the six trigonometric functions of 300°.

SOLUTION: Referring to figure 4-16, 300° is in the fourth quadrant and its reference angle is 60°. Therefore,

$$\sin 300^\circ = -\sqrt{3}/2$$

$$\cos 300^\circ = 1/2$$

$$\tan 300^\circ = -\sqrt{3}/1 = -\sqrt{3}$$

$$\cot 300^\circ = 1/-\sqrt{3} = -\sqrt{3}/3$$

$$\sec 300^\circ = 2/1 = 2$$

$$\csc 300^\circ = 2/-\sqrt{3} = -2\sqrt{3}/3$$

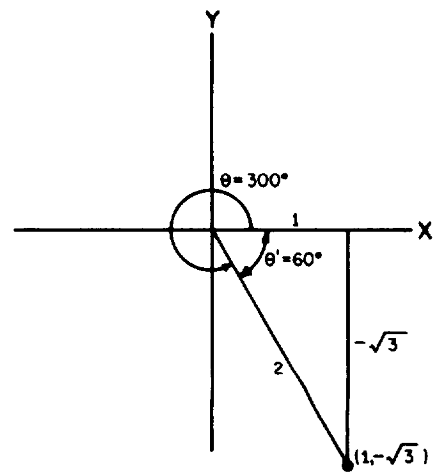

Figure 4-16.—300° angle in standard position.

45° Angles

Refer to figure 4-17. If one of the acute angles of the right triangle OXY is 45°, then the other acute angle is also 45°. Since triangle OXY is an isosceles triangle, then sides OX and XY are equal. If we let OX and XY be 1 unit long, then by the Pythagorean theorem, the length of OY is $\sqrt{2}$ units.

NOTE: This relationship is true of all 45° triangles and is not altered by the lengths of the legs. The ratio of the sides of right triangles with 45° complementary angles will always be 1, 1, and $\sqrt{2}$, where the largest value represents the length of the hypotenuse.

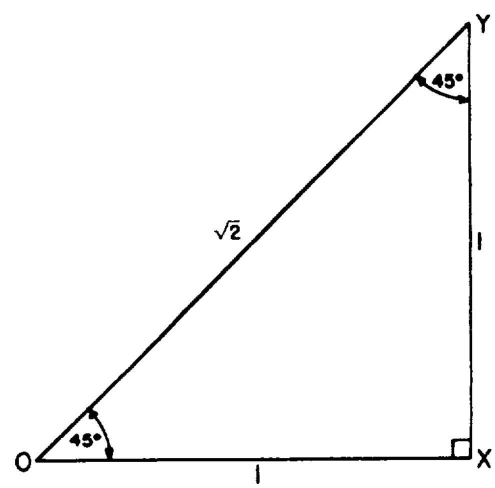

Figure 4-17.—Geometrical construction of a 45° right triangle.

Figure 4-18 shows a 45° reference triangle. From this figure we can determine the trigonometric ratios of 45°, which are also summarized in table 4-1.

EXAMPLE: Find the six trigonometric functions of 135°.

SOLUTION: Referring to figure 4-19, 135° is in the second quadrant and its reference angle is 45°. Therefore,

$$\sin 135° = 1/\sqrt{2} = \sqrt{2}/2$$

$$\cos 135° = -1/\sqrt{2} = -\sqrt{2}/2$$

$$\tan 135° = 1/-1 = -1$$

$$\cot 135° = -1/1 = -1$$

$$\sec 135° = \sqrt{2}/-1 = -\sqrt{2}$$

$$\csc 135° = \sqrt{2}/1 = \sqrt{2}$$

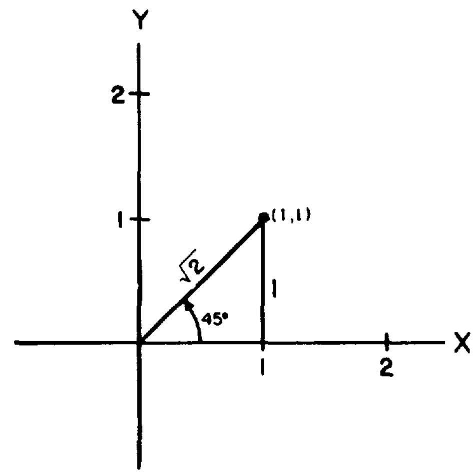

Figure 4-18.—45° reference triangle.

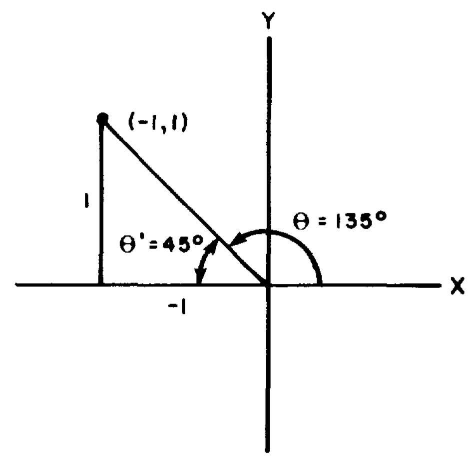

Figure 4-19.—135° angle in standard position.

QUADRANTAL ANGLES

An angle whose terminal side lies on a coordinate axis when the angle is in standard position is a *quadrantal angle.* Angles of 0°, ±90°, ±180°, and ±270° are quadrantal angles.

The trigonometric functions of the quadrantal angles are defined in the same manner as before, except for the restriction that a function is undefined when the denominator of the ratio is zero.

To derive the functions of the quandrantal angles, we choose points on the terminal sides, where $r = 1$, as shown in figure 4-20. Then either x or y is zero, and the other coordinate is either positive or negative 1.

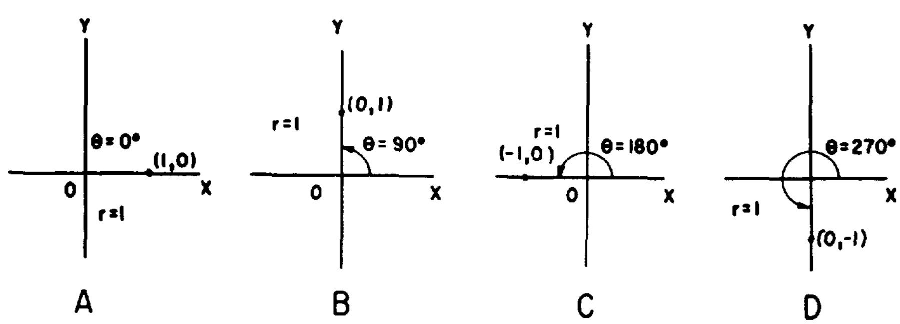

Figure 4-20.—Functions of quadrantal angles.

Table 4-2.—Functions of Quadrantal Angles

θ Deg.	θ Rad.	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	-1	0	undefined	-1	undefined
270°	$\frac{3\pi}{2}$	-1	0	undefined	0	undefined	-1

Consider view C of figure 4-20 in which $\theta = 180°$. For the point $P(-1,0)$ and $r = 1$, we have

$$\sin 180° = 0/1 = 0$$

$$\cos 180° = -1/1 = -1$$

$$\tan 180° = 0/-1 = 0$$

$$\cot 180° = -1/0 \text{ (undefined)}$$

$$\sec 180° = 1/-1 = -1$$

$$\csc 180° = 1/0 \text{ (undefined)}$$

The values of the functions of the other quadrantal angles can be found by a similar procedure and are summarized in table 4-2.

EXAMPLE: Determine the six trigonometric functions of 990°.

SOLUTION: Referring to figure 4-21, we see that 990° lies on the same quadrantal axes as 270°. Therefore, for $P(0,-1)$ and $r = 1$, we have

$$\sin 990° = -1/1 = -1$$

$$\cos 990° = 0/1 = 0$$

$$\tan 990° = -1/0 \text{ (undefined)}$$

$$\cot 990° = 0/-1 = 0$$

$$\sec 990° = 1/0 \text{ (undefined)}$$

$$\csc 990° = 1/-1 = -1$$

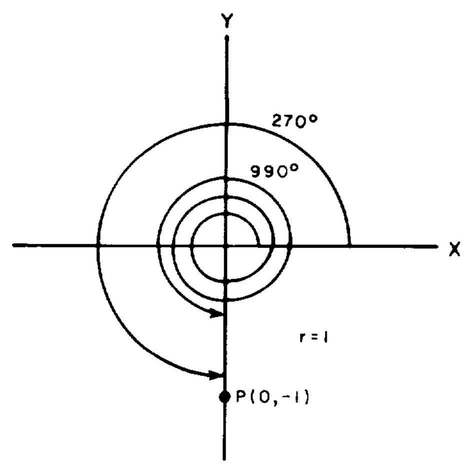

Figure 4-21.—990° angle.

PRACTICE PROBLEMS:

Without using the appendixes, determine the trigonometric functions of problems 1 through 5.

1. $\theta = 210°$

2. $\theta = 360°$

3. $\theta = 585°$

4. $\theta = -180°$

5. $\theta = -315°$

Without using the appendixes, evaluate problems 6 through 8. [Note that $\sin^2 \theta = (\sin \theta)^2$.]

6. $\sin^2 150° + \cos^2 150°$

7. $2 \sin 120° \cos 120°$

8. $\cos^2 135° - \sin^2 135°$

ANSWERS:

1. $\sin 210° = -1/2$

   $\cos 210° = -\sqrt{3}/2$

   $\tan 210° = -1/-\sqrt{3} = \sqrt{3}/3$

   $\cot 210° = -\sqrt{3}/-1 = \sqrt{3}$

   $\sec 210° = 2/-\sqrt{3} = -2\sqrt{3}/3$

   $\csc 210° = 2/-1 = -2$

2. $\sin 360° = 0/1 = 0$

   $\cos 360° = 1/1 = 1$

   $\tan 360° = 0/1 = 0$

   $\cot 360° = 1/0$ (undefined)

   $\sec 360° = 1/1 = 1$

   $\csc 360° = 1/0$ (undefined)

3. $\sin 585° = -1/\sqrt{2} = -\sqrt{2}/2$

$\cos 585° = -1/\sqrt{2} = -\sqrt{2}/2$

$\tan 585° = -1/-1 = 1$

$\cot 585° = -1/-1 = 1$

$\sec 585° = \sqrt{2}/-1 = -\sqrt{2}$

$\csc 585° = \sqrt{2}/-1 = -\sqrt{2}$

4. $\sin(-180°) = 0/1 = 0$

$\cos(-180°) = -1/1 = -1$

$\tan(-180°) = 0/-1 = 0$

$\cot(-180°) = -1/0$ (undefined)

$\sec(-180°) = 1/-1 = -1$

$\csc(-180°) = 1/0$ (undefined)

5. $\sin(-315°) = 1/\sqrt{2} = \sqrt{2}/2$

$\cos(-315°) = 1/\sqrt{2} = \sqrt{2}/2$

$\tan(-315°) = 1/1 = 1$

$\cot(-315°) = 1/1 = 1$

$\sec(-315°) = \sqrt{2}/1 = \sqrt{2}$

$\csc(-315°) = \sqrt{2}/1 = \sqrt{2}$

6. $(1/2)^2 + (-\sqrt{3}/2)^2 = 1$

7. $2(\sqrt{3}/2)(-1/2) = -\sqrt{3}/2$

8. $(-1/\sqrt{2})^2 - (1/\sqrt{2})^2 = 0$

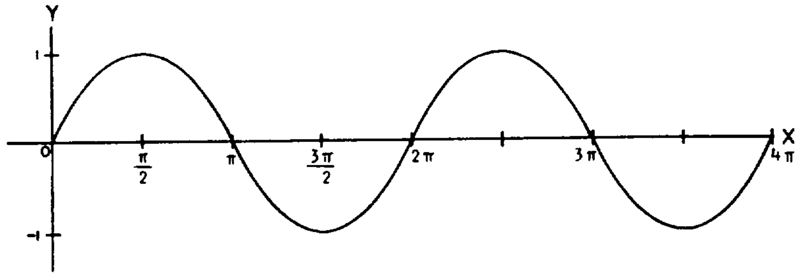

Figure 4-22.—Graph of the sine function.

PERIODS OF THE TRIGONOMETRIC FUNCTIONS

A trigonometric function of an angle is not changed in value when the angle is changed by any multiple of 360° or 2π radians. For this reason the functions are said to be *periodic.*

In the following paragraphs, the graphs of the sine, cosine, and tangent functions are discussed.

GRAPH OF THE SINE FUNCTION

Figure 4-22 shows two periods of the sine function. The graph shows that the value of the sine function varies between $+1$ and -1 and never goes beyond these limits as the angle varies. The graph also shows that the sine function increases from 0 at 0° or 0 radians to a maximum value of $+1$ at 90° or $\pi/2$ radians. It decreases back to 0 at 180° or π radians and continues to decrease to a minimum value of -1 at 270° or $3\pi/2$ radians. It then increases to a value of 0 at 360° or 2π radians. If we extend the graph (in either direction), the sine function will continue to repeat itself every 360° or 2π radians. Therefore, *the period of the sine function is 360° or 2π radians.*

GRAPH OF THE COSINE FUNCTION

The cosine function also has a period of 360° or 2π radians. Figure 4-23 shows two periods of the cosine function. The range

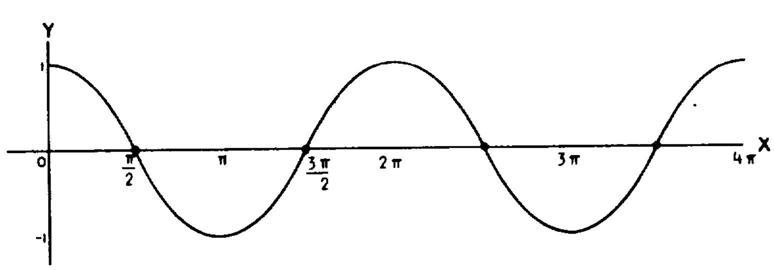

Figure 4-23.—Graph of the cosine function.

of the values the cosine function takes on also lies between $+1$ and -1. However, as seen on the graph, the cosine function decreases from 1 at 0° or 0 radians to 0 at 90° or $\pi/2$ radians and continues to decrease to a minimum value of -1 at 180° or π radians. It then increases to 0 at 270° or $3\pi/2$ radians and continues to increase to a maximum value of $+1$ at 360° or 2π radians. This completes one period of the cosine function.

GRAPH OF THE TANGENT FUNCTION

Figure 4-24 shows the graph of the tangent function from 0 radians to 2π radians. Notice that the tangent function is 0 at 0° or 0 radians and increases to positive infinity (without bounds) between 0° and 90° or 0 radians and $\pi/2$ radians. Remember that the tangent function is undefined for $90° + n(180°)$ or $\pi/2 + n\pi$, where n is any integer. The dashed vertical lines in figure 4-24 represent the undefined points. The tangent function increases from negative infinity to 0 between 90° and 180° or $\pi/2$ radians and π radians. At 180° or π radians, the tangent function is 0. The function continues to increase from 0 to positive infinity between 180° and 270° or π radians and $3\pi/2$ radians. Between 270° and 360° or $3\pi/2$ radians and 2π, it again increases from negative infinity to 0 at 360° or 2π radians. If we extend the graph (in either direction), the curve will repeat itself every 180° or π radians. Therefore, *the period of the tangent function is 180° or π radians.*

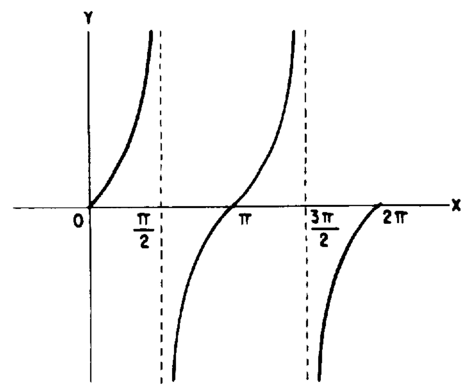

Figure 4-24.—Graph of the tangent function.

EXAMPLE: Using the graphs in figures 4-22 through 4-24, determine the values of θ, where $\sin\theta$ and $\tan\theta$ increase together, if $0 \leq \theta \leq \pi$.

SOLUTION: In figure 4-22 the sine function increases between 0 and $\pi/2$ radians for the interval of $0 \leq \theta \leq \pi$. (The sine function does not increase or decrease at points 0 or $\pi/2$.) In figure 4-24 the tangent function also increases between 0 and $\pi/2$ radians for the interval of $0 \leq \theta \leq \pi$. (The tangent function does not increase or decrease at 0 and is undefined at $\pi/2$.) Therefore, the values of θ, where $\sin\theta$ and $\tan\theta$ increase together, are $0 < \theta < \pi/2$.

PRACTICE PROBLEMS:

Use the graphs in figures 4-22 through 4-24 to answer the following problems (use appendixes II and III to verify your answers):

1. For what values of θ does $\cos\theta$ increase if $0 \leq \theta \leq \pi$?

2. For what values of θ do $\sin\theta$ and $\cos\theta$ decrease together if $0 \leq \theta \leq 2\pi$?

3. For what values of θ do $\cos\theta$ and $\tan\theta$ increase together if $\pi/2 \leq \theta \leq 3\pi/2$?

4. For what values of θ do $\sin\theta$, $\cos\theta$, and $\tan\theta$ increase together if $0 \leq \theta \leq 2\pi$?

ANSWERS:

1. None

2. $\pi/2 < \theta < \pi$

3. $\pi < \theta < 3\pi/2$

4. $3\pi/2 < \theta < 2\pi$

SUMMARY

The following are the major topics covered in this chapter:

1. **Angles in standard position:** An angle in standard position on a rectangular coordinate system has its vertex at the origin, its initial side lying along the X axis, and its terminal side lying in any of the quadrants or on one of the axes.

2. **Coterminal angles**: When two or more angles in standard position have their terminal sides located at the same position, they are said to be *coterminal.*

   For any general angle θ measured in degrees, any angle ϕ coterminal with θ can be found by

$$\phi = \theta + n(360°)$$

   where n is any integer.

3. **Definitions of the trigonometric functions:**

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{length of radius}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{length of radius}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{length of radius}}{\text{abscissa}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{length of radius}}{\text{ordinate}}$$

4. **Signs of the trigonometric ratios in the quadrant system**: All the trigonometric ratios are positive for angles in the first quadrant. Only the sine and cosecant ratios are positive in the second quadrant. Only the tangent and cotangent ratios are positive in the third quadrant. Only the cosine and secant ratios are positive in the fourth quadrant.

5. **Reference angle**: The *reference angle*, θ', for any angle, θ, in standard position is the smallest positive angle between the radius vector of θ and the X axis, such that $0° \leq \theta' \leq 90°$. In general, for any integer n,

$$\theta' = n(180°) \pm \theta$$

or

$$\theta' = n\pi \pm \theta$$

where $0 \leq \theta' \leq \pi/2$.

6. **Reference triangle:** The right triangle formed from the reference angle when you connect a point on the radius vector of the reference angle perpendicular to the X axis is called the *reference triangle*.

7. **Reduction formulas:** *Reduction formulas* are formulas used to determine the signs of the functions of any angle. They provide a means of reducing the functions of any angle to an equivalent expression for the function in terms of a positive acute angle.

8. **Quadrant I angles**: An angle in quadrant I is represented by θ.

$$\sin\theta = y/r$$

$$\cos\theta = x/r$$

$$\tan\theta = y/x$$

$$\cot\theta = x/y$$

$$\sec\theta = r/x$$

$$\csc\theta = r/y$$

9. **Quadrant II angles**: An angle in quadrant II is represented by $180 - \theta$.

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

$$\cot(180° - \theta) = -\cot\theta$$

$$\sec(180° - \theta) = -\sec\theta$$

$$\csc(180° - \theta) = \csc\theta$$

10. **Quadrant III angles**: An angle in quadrant III is represented by $180° + \theta$.

$$\sin(180° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\tan(180° + \theta) = \tan\theta$$

$$\cot(180° + \theta) = \cot\theta$$

$$\sec(180° + \theta) = -\sec\theta$$

$$\csc(180° + \theta) = -\csc\theta$$

11. **Quadrant IV angles:** An angle in quadrant IV is represented by $360° - \theta$.

$$\sin(360° - \theta) = -\sin\theta$$

$$\cos(360° - \theta) = \cos\theta$$

$$\tan(360° - \theta) = -\tan\theta$$

$$\cot(360° - \theta) = -\cot\theta$$

$$\sec(360° - \theta) = \sec\theta$$

$$\csc(360° - \theta) = -\csc\theta$$

12. **Functions of negative angles:**

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\csc(-\theta) = -\csc\theta$$

13. **Functions of coterminal angles**: For a coterminal angle in the form of

$$\theta' = n(360°) + \theta$$

where n is any integer and θ is an integral multiple of θ', the trigonometric functions of θ' are equal to those of θ.

14. **Cofunctions and complementary angles**: *Complementary angles* are angles whose sum is 90°. Two trigonometric functions that have equal values for complementary angles are called *cofunctions*.

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\tan\theta = \cot(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

$$\sec\theta = \csc(90° - \theta)$$

$$\csc\theta = \sec(90° - \theta)$$

15. **Frequently used angles**: The trigonometric functions of 30°, 60°, and 45° can be determined geometrically. The trigonometric ratios corresponding to these functions are summarized in table 4-1.

16. **Quadrantal angles**: An angle whose terminal side lies on a coordinate axis when the angle is in standard position is a *quadrantal angle*. The trigonometric ratios corresponding to the functions of the quadrantal angles are summarized in table 4-2.

17. **Periods of the trigonometric functions**: A trigonometric function of an angle is not changed in value when the angle is changed by any multiple of 360° or 2π radians. For this reason the functions are said to be *periodic*. The periods of the sine and cosine functions are 360° or 2π radians. The period of the tangent function is 180° or π radians.

ADDITIONAL PRACTICE PROBLEMS

1. Are the angles $840°$, $-240°$, and $600°$ coterminal?

2. Find the sine, cosine, and tangent of the angle θ whose radius vector passes through the point $P(\sqrt{5}, \sqrt{11})$.

3. Find the six trigonometric functions of θ if $\csc \theta = -37/35$ and $\tan \theta > 0$.

4. Find the sine, cosine, and tangent of $-4{,}010°$.

5. Express $\csc 87° 23' 13''$ as a function of its complementary angle.

6. Without using the appendixes, evaluate $\sec^2(-135°) + \cot^2(-690°) + \csc^2(-600°)$.

7. Without using the appendixes, find the six trigonometric functions of $-3{,}510°$.

8. For what values of θ do $\cos \theta$ and $\tan \theta$ both increase and $\sin \theta$ decrease together if $0 \leq \theta \leq 2\pi$?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. No

2. $\sin\theta = \sqrt{11}/4$

   $\cos\theta = \sqrt{5}/4$

   $\tan\theta = \sqrt{11}/\sqrt{5} = \sqrt{55}/5$

3. $\sin\theta = -35/37$

   $\cos\theta = -12/37$

   $\tan\theta = 35/12$

   $\cot\theta = 12/35$

   $\sec\theta = -37/12$

   $\csc\theta = -37/35$

4. $\sin\theta = -0.76604$

   $\cos\theta = 0.64279$

   $\tan\theta = -1.19175$

5. $\sec 2^\circ\, 36'\, 47''$

6. 6 1/3

7. $\sin\theta = 1$

   $\cos\theta = 0$

   $\tan\theta$ is undefined

   $\cot\theta = 0$

   $\sec\theta$ is undefined

   $\csc\theta = 1$

8. $\pi < \theta < 3\pi/2$